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A FUNCTIONAL LAW OF THE ITERATED LOGARITHM FOR DISTRIBUTIONS IN THE DOMAIN OF PARTIAL ATTRACTION OF THE NORMAL DISTRIBUTION

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A real-variable proof of a functional generalised law of the iterated logarithm due to Kesten, Kuelbs and Zinn is given, and extended to a trimmed case.

functional law of iterated logarithm * domain of partial attraction of normal * trimmed variables

1. Introduction

The purpose of this paper is to give a self contained real-variable proof of the functional form of a general law of the iterated logarithm due to Kesten [9] just as Strassen [20] gave the functional form of the law of the iterated logarithm for random variables with finite variance. This result has already been proved in essence by Kuelbs and Zinn [10] using Banach space theory. The present proof by more traditional methods gives subsidiary results which may be of interest. We also give a "trimmed" version of the result which is new.

Let X_i be iidrv's in the domain of partial attraction of the normal distribution, i.e. for which there is a sequence of integers through which the partial sums

$$S_n = X_1 + X_2 + \cdots + X_n$$

when suitably normed and centred, converge in distribution to a Gaussian r.v. Abbreviate this to $S_n \in D_p(2)$; necessary and sufficient conditions for this weak convergence property are known in terms of the distribution of X_1 and go back to Lévy in the 1930's. The remarkable result of Kesten, building on work of Heyde [8] and Rogozin [19], is that these distributions, and only these, possess generalised iterated logarithm type behaviour.

Define the random polygon

$$S_n(t) = \{(S_{k-1} - \alpha_{k-1}) + n(t-(k-1)/n)((S_k - \alpha_k) - (S_{k-1} - \alpha_{k-1}))\}/B(n)$$

for $1 \leq k \leq n$, $n \geq 1$, $(k-1)/n \leq t \leq k/n$, equivalently.

$$S_n(t) = \{(S_{[nt]} - \alpha_{[nt]}) + (nt - [nt])(X_{[nt]+1} - \alpha_{[nt]+1} + \alpha_{[nt]})\}/B(n) \qquad (1.1)$$

for $0 \leq t \leq 1$, where $S_0 = \alpha_0 = 0$, $[\cdot]$ denotes the integer part of, and α_n and $B(n)$ are sequences of real numbers, $B(n)$ being positive and nondecreasing, and $\alpha_n = nEX_1 I(|X_1| \leq B(n))$.

Let $C[0, 1]$ and $D[0, 1]$ be the spaces of continuous and right continuous functions with left limits on $[0, 1]$ with the supremum metric and let $\mathcal{K}$ be the subspace of $C[0, 1]$ defined by

$$\mathcal{K} = \left\{ f(t) \in C[0, 1]: f(0) = 0,\ f \text{ is absolutely continuous, and } \int_0^1 (\mathrm{d}f(t)/\mathrm{d}t)^2 \, \mathrm{d}t \leq 1 \right\}.$$

Our main result is

Theorem 1. $S_n \in D_p(2)$ *if and only if for some choice of* $B(n)$, *the set of almost sure limit points of* $S_n(t)$ *as a sequence in* $C[0, 1]$ *or of* $(S_{[nt]} - \alpha_{[nt]})/B(n)$ *as a sequence in* $D[0, 1]$ *equals* $\mathcal{K}$.

Our method of proof of Theorem 1 relies heavily on the techniques of Kesten, Finkelstein [7] and Chover [3]. Kesten's result, which can be deduced from Theorem 1 by the continuous mapping theorem, is that $S_n \in D_p(2)$ if and only if, for some $B(n)$,

$$-1 = \liminf_{n \to +\infty} (S_n - \alpha_n)/B(n) < \limsup (S_n - \alpha_n)/B(n) = 1 \quad \text{a.s.}$$

(Kesten actually allows a more general sequence of centring constants.) A functional law of the iterated logarithm for variables with infinite variance but finite mean has been given by Pakshirajan and Vasudeva [18], see also Weiner [22], Maller [13], [14] for related results.

Next we show the "light trimming", i.e. removing a fixed number of terms of largest modulus from S_n, does not affect behaviour of the type of Theorem 1. Here we follow the pioneering work of Feller [6] and Mori [16, 17]. For $1 \leq j \leq n$ let $m_n(j)$, be the number of X_i satisfying either

$$|X_i| > |X_j|, \quad 1 \leq i \leq n,$$

or

$$|X_i| = |X_j|, \quad 1 \leq i \leq j,$$

and let

$$X_n^{(r)} = X_j \quad \text{if } m_n(j) = r, \ 1 \leq r \leq n.$$

Thus $X_n^{(r)}$ is the term of rth largest modulus among $X_1, \ldots, X_n$, defined in a unique way in case of coincidences. Let ${}^{(r)}S_n = S_n - X_n^{(1)} - \cdots - X_n^{(r)}$ if $1 \leq r < n$ and take ${}^{(r)}S_n = 0$ for $0 \leq n \leq r$. Let ${}^{(r)}S_n(t)$ be the polygonal function obtained by interpolating linearly between the points $(k/n, ({}^{(r)}S_k - \alpha_k)/B(n))$ for $0 \leq k \leq n$.

Theorem 2. *$S_n \in D_p(2)$ if and only if for some choice of $B(n)$, ${}^{(r)}S_n(t)$ has as its set of almost sure limit points the set $\mathscr{K}$, if r is a fixed integer $\geqslant 1$.*

Applications of invariance principles like Theorems 1 and 2 abound in the literature and some examples were provided by Strassen himself. Most of these generalise immediately to our case. More recent examples, again depending on the continuous mapping theorem, have been given by Durrett and Resnick [5] (random time changes), Wellner [23] (tail results for the supremum), Barbour [1] (tail results for the sum), Vervaat [21] (the counting process $N(t) = \#(S_n \in [0, t])$). These can all be generalised to the $D_p(2)$ case under appropriate conditions, but the details are omitted here.

2. Proof of Theorem 1

We need only prove sufficiency, so let $S_n \in D_p(2)$. Following [9], there is a sequence $x_k \uparrow +\infty$ for which

$$\zeta_k = x_k^2 H(x_k)/V(x_k) \to 0 \tag{2.1}$$

where

$$H(x_k) = P(|X_1| > x_k), \qquad V(x_k) = \operatorname{Var}(X_1^k), \qquad X_i^k = X_i I(|X_i| \leqslant x_k).$$

Assume by taking a further subsequence if necessary that $\zeta_k \leqslant k^{-2/\varepsilon}$, where ε is fixed throughout the proof between 0 and $\frac{1}{4}$. Note that $\zeta_k/H(x_k) \to +\infty$, so $\zeta_k^{1/4}/H(x_k) \to +\infty$, and we can assume also that $\zeta_k^{1/4}/H(x_k)$ is strictly increasing.

Our norming sequence is defined by

$$B(n) = n^{1/2-\varepsilon}\{\zeta_k^{1/4}/H(x_k)\}^{\varepsilon}(2V(x_k)\log k)^{1/2},$$

$$[\zeta_{k-1}^{1/4}/H(x_{k-1})] < n \leqslant [\zeta_k^{1/4}/H(x_k)],$$

and it will be notationally convenient to introduce a sequence r_k defined by

$$r_k = \log_\lambda[\zeta_k^{1/4}/H(x_k)] \tag{2.2}$$

where $\lambda > 1$. Let $\lambda_r = [\lambda^r]$, the integer part of λ^r. Now $B(n)$ does not depend on λ but we have, for each $\lambda > 1$,

$$B(n) = n^{1/2-\varepsilon}\lambda_{r_k}^{\varepsilon}(2V(x_k)\log k)^{1/2}, \quad \lambda_{r_{k-1}} < n \leqslant \lambda_{r_k}. \tag{2.3}$$

Let

$$I_k = \{i\colon r_k - \varepsilon^{-1}\log_\lambda k < i \leqslant r_k\}$$

and let

$$J_k = \{i\colon r_{k-1} < i \leqslant r_k - \varepsilon^{-1}\log_\lambda k\}.$$

Define $\nu(x_k)=EX_1^k$ and let $\alpha_n=n\nu(B(n))$. Let

$$\Phi(x)=(2\pi)^{-1/2}\int_{-\infty}^{x}\mathrm{e}^{-u^2/2}\,\mathrm{d}u.$$

It will be useful to note that

$$\begin{aligned}\sum_k\sum_{r_{k-1}<r\le r_k}\lambda_r H(x_k)&\le\sum_k H(x_k)\sum_{r=0}^{r_k}\lambda^r\\&\le\sum_k\lambda^{r_k}H(x_k)/(\lambda-1)\\&\le\sum_k\zeta_k^{1/4}/(\lambda-1)\\&<+\infty\end{aligned}\tag{2.4}$$

and for k large enough (we always assume this but often omit to mention it) $r\in J_k$ implies $\lambda_{r_{k-1}}<r\le\lambda_{r_k}$, so, by (2.3),

$$\begin{aligned}\sum_k\sum_{r\in J_k}\lambda_r V(x_k)/B^2(\lambda_r)&=\tfrac12\sum_k\sum_{r\in J_k}\lambda_r^{2\varepsilon}/(\lambda_{r_k}^{2\varepsilon}\log k)\\&\le\tfrac12\sum_k\sum_{0\le r\le r_k-\varepsilon^{-1}\log k}\lambda^{2\varepsilon r}/(\lambda_{r_k}^{2\varepsilon}\log k)\\&\le\tfrac12(1/(\lambda^{2\varepsilon}-1))\sum_k(\lambda_{r_k}+1)^{2\varepsilon}/(\lambda_{r_k}^{2\varepsilon}k^2\log k)\\&<+\infty.\end{aligned}\tag{2.5}$$

Then (2.4) and (2.5) imply

$$\sup_{r_{k-1}<r\le r_k}\lambda_r H(x_k)\to0,\quad\sup_{r\in J_k}\lambda_r V(x_k)/B^2(\lambda_r)\to0\qquad(k\to+\infty).\tag{2.6}$$

To apply the method of Finkelstein [7], define

$$S_n^*=c_1(X_1+\cdots+X_n)+c_2(X_{n+1}+\cdots+X_{2n})+\cdots+c_p(X_{(p-1)n+1}+\cdots+X_{pn})\tag{2.7}$$

where $p\ge1$ is an integer and $c_1,\ldots,c_p$ are constants satisfying $c_1^2+c_2^2+\cdots+c_p^2=1$. Let $c.=c_1+c_2+\cdots+c_p$. The basic bounds required come from the following lemma.

Lemma 1. *If $t>0$, $\lambda>1$ and $a>1$ then*

$$\sum_k\sum_{r\in I_k}\sup_x|P\{S^*_{[t\lambda_r]}-c.[t\lambda_r]\nu(x_k)<x([t\lambda_r]V(x_k))^{1/2}\}-\Phi(x)|<+\infty\tag{2.8}$$

and

$$\sum_k\sum_{r_{k-1}<r\le r_k}P\left\{\sup_{1\le j\le[t\lambda_r]}|S_j^*-c.j\nu(x_k)|>at^{1/2}B(\lambda_r)\right\}<+\infty.\tag{2.9}$$

Proof. Define

$$S_n^{*k} = c_1(X_1^k + \cdots + X_n^k) + \cdots + c_p(X_{(p-1)n+1}^k + \cdots + X_{pn}^k)$$

where $X_i^k = X_i I(|X_i| \leq x_k)$. Then

$$ES_n^{*k} = c.n\nu(x_k) \quad \text{and} \quad \operatorname{Var}(S_n^{*k}) = (c_1^2 + \cdots + c_p^2)nV(x_k) = nV(x_k).$$

Writing

$$S_n^{*k} = \sum_{i=1}^{pn} c_{in} X_i^k$$

where $c_{in} = c_j$, $(j-1)n+1 \leq i \leq jn$, $1 \leq j \leq p$, represents S_n^{*k} as a sum of independent r.v.'s. By a generalisation of the Berry–Esseen theorem (e.g. Loève [11, § 21.3, p. 300])

$$|P\{S_{[t\lambda_r]}^{*k} - c.[t\lambda_r]\nu(x_k) < x([t\lambda_r]V(x_k))^{1/2}\} - \Phi(x)| \leq LL_r^k$$

where L is an absolute constant and

$$\begin{aligned} L_r^k &= \sum_{i=1}^{p[t\lambda_r]} E|c_{i[t\lambda_r]}(X_i^k - EX_i^k)|^3 / ([t\lambda_r]V(x_k))^{3/2} \\ &\leq pE|X_i^k - EX_i^k|^3 / ([t\lambda_r]^{1/2} V^{3/2}(x_k)) \end{aligned}$$

since clearly $|c_{in}| \leq 1$. Since

$$E|X_1^k - \nu(x_k)|^3 \leq 8 \max(E|X_1^k|^3, |\nu(x_k)|^3) \leq 8x_k V(x_k) \quad \text{for } k \text{ large,}$$

$r \in I_k$ implies

$$\begin{aligned} L_r^k &\leq Lt^{-1/2} x_k V(x_k) / (\lambda_r^{1/2} V^{3/2}(x_k)) \\ &= Lt^{-1/2} \{x_k^2 / (\lambda_{r_k} V(x_k))\}^{1/2} (\lambda_{r_k}/\lambda_r)^{1/2} \\ &= Lt^{-1/2} \zeta_k^{3/8} (\lambda_{r_k}/\lambda_r)^{1/2} \\ &\leq Lt^{-1/2} \zeta_k^{3/8} k^{1/2\varepsilon} \\ &\leq Lt^{-1/2} k^{-1/4\varepsilon}, \end{aligned}$$

L again denoting constants and noting that $\lambda_{r_k}/\lambda_r \leq k^{1/\varepsilon}$ when $r \in I_k$. Since $\varepsilon < \frac{1}{4}$, $\sum_k \sum_{r \in I_k} L_r^k < +\infty$, proving (2.8) with $S_{[t\lambda_r]}^{*k}$ replacing $S_{[t\lambda_r]}^*$.

For (2.9), put $x = at^{1/2}B(\lambda_r)/([t\lambda_r]V(x_k))^{1/2}$ in this to get

$$\sum_k \sum_{r \in I_k} P\{S_{[t\lambda_r]}^{*k} - c.[t\lambda_r]\nu(x_k) > at^{1/2}B(\lambda_r)\} < +\infty$$

since

$$\begin{aligned} &1 - \Phi\{at^{1/2}B(\lambda_r)/([t\lambda_r]V(x_k))^{1/2}\} \\ &\qquad \sim L \frac{([t\lambda_r]V(x_k))^{1/2}}{at^{1/2}B(\lambda_r)} \exp\{-\tfrac{1}{2}a^2 tB^2(\lambda_r)/([t\lambda_r]V(x_k))\} \end{aligned}$$

is the term of a convergent series when $a > 1$ and $r \in (r_{k-1}, r_k]$ because then $\lambda_r \in (\lambda_{r_{k-1}}, \lambda_{r_k}]$ and so

$$B^2(\lambda_r) = 2\lambda_r^{1-2\varepsilon} \lambda_{r_k}^{2\varepsilon} V(x_k) \log k \geq 2\lambda_r V(x_k) \log k. \tag{2.10}$$

Also by Chebychev's inequality

$$\sum_k \sum_{r\in J_k} P\{S^{*k}_{[t\lambda_r]} - c.[t\lambda_r]\nu(x_k) > at^{1/2}B(\lambda_r)\} \le \sum_k \sum_{r\in J_k} \mathrm{Var}(S^{*k}_{[t\lambda_r]})/(a^2tB^2(\lambda_r))$$
$$\le a^{-2}\sum_k \sum_{r\in J_k} \lambda_r V(x_k)/B^2(\lambda_r)$$
$$< +\infty$$

by (2.5). So we have

$$\sum_k \sum_{r_{k-1}<r\le r_k} P\{S^{*k}_{[t\lambda_r]} - c.[t\lambda_r]\nu(x_k) > at^{1/2}B(\lambda_r)\} < +\infty.$$

Then (2.9) follows from a minor modification of Lévy's inequality (Loève [11, § 18.1, p. 260]):

$$P\left\{ \sup_{1\le j\le[t\lambda_r]} (S^{*k}_j - ES^{*k}_j) > at^{1/2}B(\lambda_r)\right\}$$
$$\le 2P\{S^{*k}_{[t\lambda_r]} - ES^{*k}_{[t\lambda_r]} > at^{1/2}B(\lambda_r) - (2\,\mathrm{Var}\, S^{*k}_{[t\lambda_r]})^{1/2}\}$$

on recalling from (2.10) that $\mathrm{Var}\, S^{*k}_{[t\lambda_r]} = [t\lambda_r]V(x_k)$ is negligible with respect to $B^2(\lambda_r)$ when $r\in(r_{k-1}, r_k]$. Modulus signs can be introduced by replacing X_i by $-X_i$, and the truncation in the above arguments is easily removed by (2.4). So (2.8) and (2.9) are proved.

Corollary 1 to Lemma 1. *Under the previous assumptions, if* $0\le t\le 1$,

$$\limsup_{n\to+\infty} |S^*_{[nt]} - c.\alpha_{[nt]}|/B(n) \le t^{1/2} \quad a.s. \tag{2.11}$$

Proof. Given n large choose $r = r(n)$ so that $\lambda_{r-1} < n < \lambda_r$. Then choose $k = k(n)$ so that $\lambda_{r_{k-1}} \le \lambda_{r-1} < \lambda_r \le \lambda_{r_k}$ which is possible since λ_{r_k} is a subsequence of λ_r. Then

$$\frac{|S^*_{[nt]} - c.[nt]\nu(x_k)|}{B(n)} \le \sup_{1\le j\le[t\lambda_r]} \frac{|S^*_j - c.j\nu(x_k)|}{B(\lambda_{r-1})}$$
$$\le t^{1/2}\lambda^{1/2} + \delta \quad \text{a.s.}$$

if n is large enough, by (2.9) and the Borel-Cantelli Lemma (and using $\lambda_r \sim \lambda\lambda_{r-1}$). Since $\lambda > 1$ is arbitrary we can let $\lambda\downarrow 1$ and $\delta\downarrow 0$ to obtain (2.11) provided we show

$$\sup_{\lambda_{r_{k-1}}<n\le\lambda_{r_k}} \frac{|[nt]\nu(x_k) - \alpha_{[nt]}|}{B(n)} \to 0 \quad \text{as } k\to+\infty. \tag{2.12}$$

This we do as follows. Firstly if $n\in(\lambda_{r_{k-1}}, \lambda_{r_k}]$ is such that $B(n)\ge x_k$, then

$$|[nt]\nu(x_k) - [nt]\nu(B(n))| = \left|[nt]\int_{x_k\le|u|\le B(n)} u\, dF(u)\right|$$
$$\le [nt]B(n)H(x_k)$$
$$\le t^{1/2}\lambda_{r_k}H(x_k)B(n)$$
$$= o(B(n))$$

by (2.6). Next suppose $n \in (\lambda_{r_{k-1}}, \lambda_{r_k}]$ is such that $B(n) < x_k$. This implies $n \in (\lambda_{r_{k-1}}, \lambda_{r_k}/k^{1/\varepsilon}]$, because if $n \in (\lambda_{r_k}/k^{1/\varepsilon}, \lambda_{r_k}]$ then

$$\begin{aligned} B^2(n) &= n^{1-2\varepsilon}\lambda_{r_k}^{2\varepsilon}(2V(x_k)\log k) \\ &\geq 2(\lambda_{r_k}/k^{1/\varepsilon})^{1-2\varepsilon}\lambda_{r_k}^{2\varepsilon}V(x_k)\log k \\ &= 2\lambda_{r_k}k^{-(1-2\varepsilon)/\varepsilon}V(x_k)\log k \\ &= 2\zeta_k^{1/4}(V(x_k)/x_k^2H(x_k))x_k^2k^{-(1-2\varepsilon)/\varepsilon}\log k \\ &= 2\zeta_k^{-3/4}x_k^2k^{-(1-2\varepsilon)/\varepsilon}\log k \\ &\geq 2k^{3/2\varepsilon}x_k^2k^{-(1-2\varepsilon)/\varepsilon}\log k \\ &= 2k^{2+1/2\varepsilon}x_k^2\log k \end{aligned}$$

so $B(n) > x_k$ for k large, a contradiction. When $n \in (\lambda_{r_{k-1}}, \lambda_{r_k}/k^{1/\varepsilon}]$ then

$$r = [\log_\lambda(n)] \leq r_k - (\log_\lambda k)/\varepsilon \in J_k,$$

so, by the Cauchy-Schwarz inequality,

$$\begin{aligned} |[nt]\nu(x_k) - [nt]\nu(B(n))|^2 &\leq t^2n^2\left|\int_{B(n)\leq|u|\leq x_k} u\,\mathrm{d}F(u)\right|^2 \\ &\leq t^2n^2H(B(n))V(x_k) \\ &= t^2nH(B(n))(nV(x_k)/B^2(n))B^2(n) \\ &\leq t^2nH(B(n))B^2(n)\sup_{r\in J_k}(\lambda_rV(x_k)/B^2(\lambda_r)) \\ &= \mathrm{o}(B^2(n))nH(B(n)) \end{aligned}$$

by (2.6). Now $nH(B(n)) \to 0$, in fact $(S_n - \alpha_n)/B(n) \xrightarrow{\mathrm{P}} 0$, because by a symmetrised version of (2.9) applied with $c_1 = 1$ we deduce $S_n^s = \mathrm{O}(\mathrm{B(n)})$ a.s. by standard arguments. From Kesten [9, Theorem 7] this implies $(S_n - \mathrm{med}(S_n))/B(n) \xrightarrow{\mathrm{P}} 0$, equivalently $(S_n - \alpha_n)/B(n) \xrightarrow{\mathrm{P}} 0$.

So far we've proved that (2.12) holds with $[nt]\nu(B(n))$ in place of $\alpha_{[nt]}$. But, for $0 \leq t \leq 1$,

$$\begin{aligned} |[nt]\nu(B(n)) - \alpha_{[nt]}| &= [nt]\left|\int_{B[nt]\leq|u|\leq B(n)} u\,\mathrm{d}F(u)\right| \\ &\leq [nt]H(B[nt])B(n) \\ &= \mathrm{o}(B(n)), \end{aligned}$$

so (2.12) itself holds.

Corollary 2 to Lemma 1. *$S_n(t)$ is equicontinuous on* $[0, 1]$.

Proof. We have to show

$$\lim_{\delta\to 0}\limsup_{n\to+\infty}\sup_{0<t'-t<\delta}|S_n(t') - S_n(t)| = 0 \quad \text{a.s.},$$

which by the linear nature of $S_n(t)$ is equivalent to

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{\substack{0<k-j\le[n\delta]\\0\le j<k\le n}}\left|\sum_{i=j+1}^{k}X_i-\alpha_k+\alpha_j\right|\bigg/B(n)=0\quad\text{a.s.}\tag{2.13}$$

since it suffices to consider points t, t' of the form $t=j/n<k/n=t'$.

Consider first a symmetrised version of (2.13), where X_i is replaced by X_i^s which has the distribution of the difference of two independent copies of X_i, and α_k and α_j are omitted. Now

$$\sup_{\substack{0<k-j\le[n\delta]\\0\le j<k\le n}}\left|\sum_{i=j+1}^{k}X_i^s\right|\le\sup_{1\le m\le[1/\delta]}\sup_{j,k\in I(n,m)}\left|\sum_{i=j+1}^{k}X_i^s\right|$$

$$\le 2\sup_{1\le m\le[1/\delta]}\sup_{k\in I(n,m)}\left|\sum_{i=(m-1)[n\delta]+1}^{k}X_i^s\right|\tag{2.14}$$

where $I(n,m)=\{i:(m-1)[n\delta]<i\le(m+1)[n\delta]\}$, $1\le m<[1/\delta]$, is a partition of $[1,n]$ into overlapping sets. To prove the required convergence we go to a geometric subsequence and show

$$\sum_n P\left\{\sup_{1\le m\le[1/\delta]}\sup_{k\in I(\lambda_n,m)}\left|\sum_{(m-1)[\lambda_n\delta]+1}^{k}X_i^s\right|>4(\delta\lambda)^{1/2}B(\lambda_{n-1})\right\}<+\infty\tag{2.15}$$

which is implied by

$$\sum_n P\left\{\sup_{k\in I(\lambda_n,m)}\left|\sum_{(m-1)[\lambda_n\delta]+1}^{k}X_i^s\right|>4(\delta\lambda)^{1/2}B(\lambda_{n-1})\right\}<+\infty$$

which holds by stationarity if

$$\sum P\left\{\sup_{k\in I(\lambda_n,m)-(m-1)[\lambda_n\delta]}\left|\sum_{i=1}^{k}X_i^s\right|>4(\delta\lambda)^{1/2}B(\lambda_{n-1})\right\}<+\infty$$

for each $m\le[1/\delta]$. The last series is

$$\le\sum P\left\{\sup_{1\le k\le 2[\lambda_n\delta]}\left|\sum_{i=1}^{k}X_i^s\right|>4(\delta\lambda)^{1/2}B(\lambda_{n-1})\right\}$$

and this is finite by a symmetrised version of (2.9). Thus (2.15) holds.

Now (2.14), (2.15) and the strong symmetrisation inequalities [11, § 18.1, p. 259] imply

$$\sum_n P\left\{\sup_{\substack{0<k-j\le[\lambda_n\delta]\\0\le j<k\le\lambda_n}}\left|\sum_{j+1}^{k}X_i-m_{k-j}\right|>4(\delta\lambda)^{1/2}B(\lambda_{n-1})\right\}<+\infty$$

where $m_{k-j}=\text{median}\,(\sum_{j+1}^{k}X_i)=\text{median}\,(S_{k-j})$. By the Borel-Cantelli lemma we now have

$$\limsup_{n}\sup_{\substack{0<k-j\le[\lambda_n\delta]\\0\le j<k\le\lambda_n}}\left|\sum_{j+1}^{k}X_i-m_{k-j}\right|\bigg/B(\lambda_{n-1})\le 4(\delta\lambda)^{1/2}\quad\text{a.s.}$$

and standard arguments show that

$$\limsup_{n} \sup_{\substack{0<k-j\leq[n\delta]\\0\leq j<k\leq n}} \left|\sum_{j+1}^{k} X_i - m_{k-j}\right| \Big/ B(n) \leq 4(\delta\lambda)^{1/2} \quad \text{a.s.}$$

So to prove (2.13) we need only replace m_{k-j} with $\alpha_k - \alpha_j$ in this. First note that

$$\sup_{0\leq k\leq n} |m_n - m_k - m_{n-k}| = \mathrm{o}(B(n)) \quad (n\to+\infty); \tag{2.16}$$

if not, there would be a sequence $k=k(n)\leq n$ for which

$$|m_n - m_k - m_{n-k}| \geq \delta B(n)$$

for some $\delta>0$ and $n\geq n_0(\delta)$. But then

$$\begin{aligned} P\{|S_{n-k} - m_{n-k}| \leq \tfrac{1}{2}\delta B(n)\} &= P\{|S_n - S_k - m_n + m_k + (m_n - m_k - m_{n-k})| \leq \tfrac{1}{2}\delta B(n)\} \\ &\leq P\{|(S_n - m_n) - (S_k - m_k)| > \tfrac{1}{2}\delta B(n)\} \\ &\leq P\{|S_n - m_n| > \tfrac{1}{4}\delta B(n)\} + P\{|S_k - m_k| > \tfrac{1}{4}\delta B(n)\} \\ &\to 0 \end{aligned}$$

since $(S_n - m_n)/B(n) \xrightarrow{\mathrm{P}} 0$. But this also means

$$P\{|S_{n-k} - m_{n-k}| \leq \tfrac{1}{2}\delta B(n)\} \geq P\left\{\sup_{1\leq j\leq n} |S_j - m_j| \leq \tfrac{1}{2}\delta B(n)\right\} \to 1$$

giving a contradiction. Thus (2.16) holds. But now choose k_0 so large that

$$\sup_{0\leq j\leq k} |m_k - m_j - m_{k-j}| \leq \delta B(k) \quad \text{if } k\geq k_0.$$

Then, if $n\geq k_0$,

$$\begin{aligned} \sup_{0\leq j\leq k\leq n} |m_k - m_j - m_{k-j}| &\leq \sup\left\{\sup_{k_0\leq k\leq n} \delta B(k), \sup_{\substack{0\leq j\leq k_0\\0\leq k\leq k_0}} |m_k - m_j - m_{k-j}|\right\} \\ &\leq \sup\{\delta B(n), \text{constant}\} \\ &= \mathrm{o}(B(n)). \end{aligned}$$

So we can replace m_{k-j} with $m_k - m_j$. But also, $(m_n - \alpha_n)/B(n)\to 0$, so

$$\sup_{0\leq j\leq k\leq n} |(m_k - m_j) - (\alpha_k - \alpha_j)| \leq 2 \sup_{0\leq k\leq n} |m_k - \alpha_k| = \mathrm{o}(B(n))$$

and we can replace $m_k - m_j$ with $\alpha_k - \alpha_j$. This proves (2.13) and the Corollary.

Lemma 2. *$(S^*_{[nt]} - c.\alpha_{[nt]})/B(n)$ has as its set of almost sure limit points the interval $[-t^{1/2}, t^{1/2}]$.*

Proof. In view of Corollary 1 to Lemma 1 it suffices to show that $(S^*_{[nt]} - c.\alpha_{[nt]})/B(n)$ is recurrent at each point of $(-t^{1/2}, t^{1/2})$; since the set of almost sure limit points is closed this will prove the assertion. Taking the subsequence λ_{r_k} of n it thus suffices to show that for $\delta > 0$ and $|a| < 1$,

$$P\left\{\left|\frac{S^*_{\beta_k} - c.\beta_k \nu(x_k)}{B(\lambda_{r_k})} - at^{1/2}\right| < 3\delta \text{ io}\right\} = 1$$

where to simplify notation $\beta_k = [t\lambda_{r_k}]$. Fix $m \geq 1$. We can write

$$S^*_{\beta_k} = \sum_{q=1}^{m} \left\{c_1 \sum_{i\in Q_1} X_i + \cdots + c_p \sum_{i \in Q_P} X_i\right\}$$

where, for $1 \leq j \leq p$,

$$Q_j = \{i\colon (q-1)\beta_k/m + (j-1)\beta_k < i \leq q\beta_k/m + (j-1)\beta_k\},$$

so

$$S^*_{\beta_k} - c.\beta_k\nu(x_k) = \sum_{q=1}^{m}\sum_{j=1}^{p} c_j\left\{\sum_{i\in Q_j} X_i - \beta_k\nu(x_k)/m\right\}. \tag{2.17}$$

We can ignore the term for $q = 1$ in this as follows. Choose m so large that $(p/m)^{1/2} \leq \delta$; we have $\sum_{j=1}^{p} |c_j| \leq p^{1/2}$ by Cauchy-Schwartz and for $1 \leq j \leq p$, by stationarity

$$\sum_k P\left\{\left|\sum_{i\in Q_j} X_i - \beta_k\nu(x_k)/m\right| > bB(\lambda_{r_k})/m^{1/2}\right\}$$

$$= \sum_k P\{|S_{\beta_k/m} - \beta_k\nu(x_k)/m| > bB(\lambda_{r_k})/m^{1/2}\}$$

$$< +\infty$$

when $b > 1$ by (2.9) of Lemma 1. So

$$\limsup_k \left|\sum_{i\in Q_j} X_i - \beta_k\nu(x_k)/m\right| \Big/ B(\lambda_{r_k}) \leq m^{-1/2} \quad \text{a.s.}$$

and the term for $q = 1$ in (2.17) is $\leq (p/m)^{1/2} < \delta$ a.s. So, if k is large enough,

$$\left|\frac{S^*_{\beta_k} - c.\beta_k\nu(x_k)}{B(\lambda_{r_k})} - at^{1/2}\right|$$

$$\leq \sum_{q=2}^{m}\left|\sum_{j=1}^{p} c_j \frac{\{\sum_{i\in Q_j} X_i - \beta_k\nu(x_k)/m\}}{B(\lambda_{r_k})} - \frac{at^{1/2}}{m}\right| + \delta + 1/m + \mathrm{o}(1)$$

$$\leq m \sup_{2\leq q\leq m}\left|\sum_{j=1}^{p} c_j \frac{\{\sum_{i\in Q_j} X_i - \beta_k\nu(x_k)/m\}}{B(\lambda_{r_k})} - \frac{at^{1/2}}{m}\right| + 2\delta$$

almost surely. So it will suffice to show

$$P\bigcap_{q=2}^{m}\left\{\left|\sum_{j=1}^{p} c_j \frac{\{\sum_{i\in Q_j} X_i - \beta_k\nu(x_k)/m\}}{B(\lambda_{r_k})} - \frac{at^{1/2}}{m}\right| < \frac{\delta}{m} \text{ io}\right\} = 1. \tag{2.18}$$

Now for $2 \leq q \leq m$ the integers in $Q_1 \cup \cdots \cup Q_p$ for k are disjoint from those for $k+1$; because the integers in Q_j are

$$\leq (j-1)\beta_k + q\beta_k/m \leq (p-1)\beta_k + \beta_k = p\beta_k \quad \text{for } k,$$

while for $k+1$ they are

$$> (q-1)\beta_{k+1}/m \geq \beta_{k+1}/m \quad \text{if } q \geq 2.$$

And since $\lambda_{r_{k+1}}/\lambda_{r_k} \to +\infty$ we can assume

$$\beta_{k+1}/m = t\lambda_{r_{k+1}}/m > tp\lambda_{r_k} = p\beta_k$$

if k is large enough for $\lambda_{r_{k+1}}/\lambda_{r_k} > pm$. Thus the events in (2.18) are independent in k so by the converse to the Borel-Cantelli Lemma it will suffice to show

$$\sum_k P \bigcap_{q=2}^{m} \left\{ \left| \sum_{j=1}^{p} c_j \frac{\{\sum_{i \in Q_j} X_i - \beta_k \nu(x_k)/m\}}{B(\lambda_{r_k})} - \frac{at^{1/2}}{m} \right| < \frac{\delta}{m} \right\} = +\infty.$$

But the events in this are independent in q since they involve disjoint blocks of the X_i. So we need

$$\sum_k \prod_{q=2}^{m} P \left\{ \left| \sum_{j=1}^{p} c_j \frac{\{\sum_{i \in Q_j} X_i - \beta_k \nu(x_k)/m\}}{B(\lambda_{r_k})} - \frac{at^{1/2}}{m} \right| < \frac{\delta}{m} \right\} = +\infty.$$

By independence and stationarity we can shift the indices in Q_j to the sets

$$Q_j^* = \{i\colon (j-1)\beta_k/m < i \leq j\beta_k/m\}, \quad 1 \leq j \leq p.$$

But then

$$\sum_{j=1}^{p} c_j \sum_{i \in Q_j^*} X_i - \beta_k \nu(x_k)/m = S^*_{\beta_k/m} - c.\beta_k \nu(x_k)/m$$

so we have to prove

$$\sum P^{m-1} \left\{ \left| \frac{S^*_{\beta_k/m} - c.\beta_k \nu(x_k)/m}{B(\lambda_{r_k})} - \frac{at^{1/2}}{m} \right| < \frac{\delta}{m} \right\} = +\infty.$$

By (2.8) of Lemma 1, sufficient for this is the divergence of

$$\begin{aligned}
&\sum \{\Phi((at^{1/2} + \delta)\gamma_k/m) - \Phi((at^{1/2} - \delta)\gamma_k/m)\}^{m-1} \\
&\quad = \sum \left\{ \int_I e^{-x^2/2}\, dx \right\}^{m-1} \Big/ (2\pi)^{(m-1)/2} \\
&\quad \geq L \sum \{2\delta\gamma_k \exp(-\tfrac{1}{2}(at^{1/2} + \delta)^2 \gamma_k^2/m^2)\}^{m-1} \\
&\quad \sim L \sum (\log k)^{(m-1)} \exp(-(at^{1/2} + \delta)^2 (m-1) t^{-1} m^{-1} \log k)
\end{aligned}$$

where L denotes constants, I is the interval $[at^{1/2} \pm \delta]\gamma_k/m$ and

$$\begin{aligned}
\gamma_k^2 &= B^2(\lambda_{r_k})/([t\lambda_{r_k}] V(x_k)/m) \\
&= 2t^{-1} m \log k.
\end{aligned}$$

The last series diverges if $|a| < 1$ and δ is small enough, which completes the proof of Lemma 2.

We can now prove Theorem 1 by following the ideas of [7] (see also [4, pp. 37-40]).

Let $\mathscr{L}_p$ be the functions $f \in C[0,1]$ with $f(0)=0$ and f being linear on the subintervals $[0,1/p],\ldots,[(p-1)/p,1]$, $p \geq 1$. Define the linear approximation $S_n^p(t)$ to $S_n(t)$ by

$$S_n^p(t)=S_n((k-1)/p)+p(t-(k-1)/p)(S_n(k/p)-S_n((k-1)/p))$$

for $(k-1)/p \leq t \leq k/p$, $1 \leq k \leq p$. Define the 1-1, bicontinuous map $\mathscr{V}: \mathscr{L}_p \to \mathbb{R}^p$ to be the (column) p-vector with components

$$(f(1/p)-f(0),\ldots,f(1)-f((p-1)/p)).$$

Let Z_n be the (column) p-vector with components

$$((S_{[n/p]}-\alpha_{[n/p]})-(S_0-\alpha_0),\ldots,(S_n-\alpha_n)-(S_{[n(p-1)/p]}-\alpha_{[n(p-1)/p]})).$$

Since, for any constants $c_1,\ldots,c_p$ with $c_1^2+\cdots+c_p^2=1$,

$$\begin{aligned}(c_1\cdots c_p)Z_n &= S^*_{[n/p]}-c_1\alpha_{[n/p]}-c_2(\alpha_{[2n/p]}-\alpha_{[n/p]})-\cdots\\ &\quad -c_p(\alpha_n-\alpha_{[n(p-1)/p]})\\ &= S^*_{[n/p]}-c.\alpha_{[n/p]}-c_2(\alpha_{[2n/p]}-\alpha_{[n/p]}-\alpha_{[n/p]})-\cdots\\ &\quad -c_p(\alpha_n-\alpha_{[(n-1)p/p]}-\alpha_{[n/p]})\end{aligned}$$

and since the extra centring terms are $o(B(n))$ by (2.16) (with m_n replaced by α_n, as is permissible), we see from Lemma 2 that $Z_n/B(n)$ has limit points $C_p=$ sphere of radius $1/p^{1/2}$ in $\mathbb{R}^p$.

Now by (1.2), $\mathscr{V}(S_n^p)=Z_n+$terms of order

$$\sup_{0\leq t<1}|X_{[nt]+1}|/B(n) \quad\text{and}\quad \sup_{0\leq t<1}|\alpha_{[nt]}-\alpha_{[nt]+1}|/B(n).$$

But

$$\sup_{0\leq t<1}|X_{[nt]+1}|=\sup_{1\leq k\leq n}|X_k|=o(B(n)) \text{ if } X_n/B(n)\to 0 \quad \text{a.s.}$$

and this follows because, as we showed earlier, $S_n^s/B(n)$ (the symmetrised sum) is bounded almost surely, so $\sum H(\delta B(n))<+\infty$ for some $\delta>0$, hence for every $\delta>0$ since $B(n)/n^{1/2-\varepsilon}$ is nondecreasing. So indeed $X_n/B(n)\to 0$ a.s., and similarly

$$\sup_{0\leq t<1}|\alpha_{[nt]}-\alpha_{[nt]+1}|=\sup_{1\leq k\leq n}|\alpha_k-\alpha_{k-1}|=o(B(n))$$

since $(S_n-\alpha_n)/B(n)\xrightarrow{\text{P}}0$ implies $(\alpha_n-\alpha_{n-1})/B(n)\to 0$.

Thus $\mathscr{V}(S_n^p)$ has limit points C_p, so S_n^p has limit points $\mathscr{V}^{-1}(C_p)=\mathscr{L}_p\cap\mathscr{K}$. Since $S_n(\cdot)$ can be uniformly approximated by $S_n^p(\cdot)$, equicontinuity (Corollary 2 to Lemma 1) shows that $S_n(\cdot)$ has limit points $\mathscr{K}$. From (1.1) and the above working this also means $(S_{[nt]}-\alpha_{[nt]})/B(n)$ has limit points $\mathscr{K}$.

3. Proof of Theorem 2

If $S_n \in D_p(2)$, then $S_n(t)$ has a.s. limit points $\mathscr{K}$ and, as we showed earlier $\sum H(\delta B(n)) < +\infty$ for $\delta > 0$, so $X_n^{(1)}/B(n) \to 0$ a.s. and ${}^{(r)}S_n(t)$ has the same limit points as $S_n(t)$.

Conversely suppose $\limsup |{}^{(r)}S_n - \alpha_n|/B(n) < +\infty$ a.s. and $S_n \notin D_p(2)$. We show then that $({}^{(r)}S_n - \alpha_n)/B(n) \to 0$ a.s., generalising [8] and [19], and so obtain a contradiction. To prove this, follow the argument of Mori [17, p. 166] to deduce successively that

$$\limsup |{}^{(r)}S_{n+1} - {}^{(r)}S_n|/B(n) < +\infty \quad \text{a.s.}, \limsup |X_n^{(r+1)}|/B(n) < +\infty \quad \text{a.s.}$$

and so $\sum n^r H^{r+1}(\delta B(n)) < +\infty$ for some $\delta > 0$. But $S_n \notin D_p(2)$ means $\limsup_x H(\delta x)/H(x) < +\infty$ for $0 < \delta \leqslant 1$, so

$$\sum n^r H^{r+1}(\delta B(n)) < +\infty \quad \text{for all } \delta > 0. \tag{3.1}$$

Define, for $\varepsilon > 0$ (not related to the ε in the proof of Theorem 1),

$$S_n^j(\varepsilon) = \sum_{i=1}^{n} X_i I(|X_i| \leqslant \varepsilon B(\lambda_{j-1})), \quad n \geqslant 1,\ j > 1,$$

where again we use the geometric subsequence $\lambda_j = [\lambda^j]$, $\lambda > 1$. Write

$${}^{(r)}S_n - ES_n^j(\varepsilon) = S_n^j(\varepsilon) - ES_n^j(\varepsilon) - \sum_{m=1}^{r} X_n^{(m)} I(|X_n^{(m)}| \leqslant \varepsilon B(\lambda_{j-1}))$$

$$+ \sum_{i=1}^{n}{}^{*} X_i I(|X_i| > \varepsilon B(\lambda_{j-1})),$$

where $\sum_{i=1}^{*n}$ denotes summation over $1 \leqslant i \leqslant n$ with terms corresponding to $X_n^{(1)}, \ldots, X_n^{(r)}$ omitted.

For $\sum^*$, note that

$$\sum_j P\left\{ \sup_{1 \leqslant k \leqslant \lambda_j} \left| \sum_{i=1}^{k}{}^{*} X_i I(|X_i| > \varepsilon B(\lambda_{j-1})) \right| > 0 \right\}$$

$$\leqslant \sum_j P \bigcup_{k=1}^{\lambda_j} \{|X_i| > \varepsilon B(\lambda_{j-1}) \text{ for some } i \leqslant k,\ X_i \neq X_k^{(1)}, \ldots, X_k^{(r)}\}$$

$$\leqslant \sum_j P \bigcup_{k=1}^{\lambda_j} \{|X_k^{(r+1)}| > \varepsilon B(\lambda_{j-1})\}$$

$$\leqslant \sum_j P\{|X_{\lambda_j}^{(r+1)}| > \varepsilon B(\lambda_{j-1})\}$$

$$\sim \sum_j \{\lambda_j H(\varepsilon B(\lambda_{j-1}))\}^{r+1} < +\infty$$

by (3.1). For $\sum_{m=1}^{r}$, use

$$P\left\{\sup_{1\le k\le \lambda_j}\left|\sum_{m=1}^{r} X_k^{(m)} I(|X_k^{(m)}|\le \varepsilon B(\lambda_{j-1}))\right| > r\varepsilon B(\lambda_{j-1})\right\}$$

$$\le P\left\{\sup_{1\le m\le r} |X_k^{(m)}| I(|X_k^{(m)}|\le \varepsilon B(\lambda_{j-1})) > \varepsilon B(\lambda_{j-1}) \text{ for some } k\le \lambda_j\right\}$$

$$=0.$$

It remains to deal with $S_n^j(\varepsilon)-ES_n^j(\varepsilon)$. Now Bennett [2, inequality 8b] proves that if Y_i are independent r.v.'s, $|Y_i-EY_i|\le M$ and $s_n^2=\sum_{i=1}^n \mathrm{Var}(Y_i)$ then for $t>0$,

$$P\left\{\left|\sum_{i=1}^{n}(Y_i-EY_i)\right|>t\right\}\le 2\exp\{-t[(1+s_n^2/Mt)\log(1+tM/s_n^2)-1]/M\}$$

and the RHS of this is

$$\le 2\exp\{-t[\log(1+tM/s_n^2)-1]/M\}$$
$$=2\,\mathrm{e}^{t/M}(1+tM/s_n^2)^{-t/M}$$
$$\le 2\,\mathrm{e}^{t/M}(s_n^2/tM)^{t/M}.$$

Applying this with $Y_i=X_iI(|X_i|\le \varepsilon B(\lambda_{j-1}))$,

$$M=\sup|X_iI(|X_i|\le \varepsilon B(\lambda_{j-1}))-EX_iI(|X_i|\le \varepsilon B(\lambda_{j-1}))|$$
$$\le 2\varepsilon B(\lambda_{j-1}),$$
$$t=2(r+1)\varepsilon B(\lambda_{j-1})$$

and

$$s_n^2=\sum_{i=1}^{n}\mathrm{Var}\,X_iI(|X_i|\le \varepsilon B(\lambda_{j-1}))=nV(\varepsilon B(\lambda_{j-1}))$$

gives

$$\sum P\{|S_{\lambda_j}^j(\varepsilon)-ES_{\lambda_j}^j(\varepsilon)|>2(r+1)\varepsilon B(\lambda_{j-1})\}$$
$$\le 2\,\mathrm{e}^{r+1}\sum\left\{\frac{\lambda_j V(\varepsilon B(\lambda_{j-1}))}{4(r+1)\varepsilon^2 B^2(\lambda_{j-1})}\right\}^{r+1}$$
$$\le L\sum\{\lambda_{j-1}H(\varepsilon B(\lambda_{j-1}))\}^{r+1}$$

using $\liminf_{x\to+\infty} x^2H(x)/V(x)>0$ which follows since $S_n\notin D_p(2)$.

Thus by (3.1)

$$\sum P\{|S_{\lambda_j}^j(\varepsilon)-ES_{\lambda_j}^j(\varepsilon)|>2(r+1)\varepsilon B(\lambda_{j-1})\}<+\infty$$

and, by the version of Lévy's inequality used for (2.9),

$$\sum P\left\{\sup_{1\le k\le \lambda_j}|S_k^j(\varepsilon)-ES_k^j(\varepsilon)|>3(r+1)\varepsilon B(\lambda_{j-1})\right\}<+\infty$$

on noting that

$$\frac{B^2(\lambda_{j-1})}{2\,\mathrm{Var}\,S_{\lambda_j}(\varepsilon)} = \frac{B^2(\lambda_{j-1})}{2\lambda_j V(\varepsilon B(\lambda_{j-1}))} \to +\infty$$

because $nB^{-2}(n)\,V(\varepsilon B(n)) = O(nH(\varepsilon B(n)) \to 0$ by (3.1).

These estimates together prove that

$$\sum P\left\{\sup_{1\le k\le \lambda_j} |{}^{(r)}S_k - ES_k^j(\varepsilon)| > 3(r+1)\varepsilon B(\lambda_{j-1})\right\} < +\infty$$

and, so by Borel-Cantelli,

$$\limsup_j \sup_{1\le k\le \lambda_j} |{}^{(r)}S_k - ES_k^j(\varepsilon)|/B(\lambda_{j-1}) \le 3(r+1)\varepsilon \quad \text{a.s.}$$

If n is large and j is chosen so that $\lambda_{j-1} < n \le \lambda_j$,

$$\frac{|{}^{(r)}S_n - n\nu(\varepsilon B_n)|}{B(n)} \le \sup_{1\le n\le \lambda_j} \frac{|{}^{(r)}S_n - ES_n^j(\varepsilon)|}{B(\lambda_{j-1})}$$

and since $(n\nu(\varepsilon B_n) - \alpha_n)/B(n) \to 0$ as a result of $nH(\varepsilon B(n)) \to 0$, we have $({}^{(r)}S_n - \alpha_n)/B(n) \to 0$ a.s., completing the proof.

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